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HOMOGENIZED CONSTITUTIVE LAW FOR A PARTIALLY COHESIVE COMPOSITE MATERIAL

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Abstract—The homogenization method is used to analyse the equivalent behavior of an elastic composite material when a tangential slip is allowed on the interface of the components. Asymptotic expansions lead to the definition of the elastic constitutive law of the homogeneous equivalent material. Numerical computations using a finite element method are performed on a fiber reinforced material. Results show the existence of a limit slip coefficient beyond which the stiffness of the material rapidly decreases.

1. INTRODUCTION

When the dimensions of the components of a mixture are much smaller than the structural dimensions, continuum models in which the inhomogeneities are "smoothed out" often suffice to describe the motion of the composite. Such examples are found in the use of the effective modulus theory to describe laminated or fiber-reinforced composite[1]. Another approach, the effective stiffness theory, using energy methods and developed by Achenbach *et al.*[2] is able to predict dispersion for waves. Nevertheless these methods account for particular geometry, such as laminated or fiber-reinforced composites. The homogenized method proposed herein, founded on a two scales asymptotic expansion, holds in any case of geometry. Provided the assumption that the internal structure of an elastic composite material is periodic and moreover that the period is small compared with the size of the entire structure, the method[3] allows to determine the elastic properties of an equivalent homogeneous medium. It has been successfully applied[4] to the survey of elastic fiber reinforced materials which are now currently used in various industries.

The goal of this work is to investigate the concept of damage in composite materials. We consider the effect of a tangential slip at the interface of the components (fiber displacements in a matrix for instance). A linear law in terms of a scalar coefficient k is used to describe this slip phenomenon.

Section 2 presents the governing equations of the structure and the mathematical formulation of the problem. Section 3 is devoted to the application of the homogenization method. Two kinds of investigations can be performed in terms of homogenization theory: the energy method[5] and the asymptotic expansions method. The energy method allows to state many mathematical theorems about convergence of the expansions and existence of the solutions. However, we choose to exhibit, in this paper the asymptotic expansions method which is particularly convenient for the preliminary studies of a periodic structure. In Section 4, the equivalent constitutive law is derived. The equivalent medium is an homogeneous, non isotropic material, and its elastic moduli depend on the parameter k involved in the slip law. The numerical application to an elastic fiber reinforced material is developed in Section 5. With additional singular elements along the interface of the components, the finite element method allows to compute the elastic moduli mentioned above. The behavior of the equivalent material is considered for various matrix–fiber ratios. The results show the existence of two limit values of a damage coefficient D expressed in term of k. If D is lower than the smaller one, the material behaves like the classical composite material in which no slip is allowed between the components. Otherwise, the elastic properties of the equivalent material rapidly decrease. When D is greater than the higher limit value, a damaged state of the material is reached.

2. THE GOVERNING EQUATIONS

We consider a composite material with two components. The first one, denoted inclusion, is an elastic material with generally high performances (such as glass fibers or carbon fibers). The remaining part of the material, the matrix, is filled up with a low performance material (epoxy for instance). The slip conditions are defined along the interface between the components.

However, one can point out that the following theoretical results extend to the hybrid composite case.

(a) *Notations*

We consider an elastic material with a periodic structure in the space variable x. More precisely, the open subset Ω of $\mathbf{R}^3$ filled with the unstressed material, can be covered by a set of periods P_i, each of these periods being the homothetic in ratio $\epsilon(\epsilon > 0$, given) of a basic period $Y =]0, Y_2[\times]0, Y_2[\times]0, Y_3[$. See Fig. 1.

Each period Y contains two subregions: Y_2 which corresponds to the inclusions (in finite number) and Y_1 which corresponds to the matrix. Γ is the boundary between Y_1 and Y_2.

Let Ω_1^ϵ denote the connected open set filled with the elastic matrix, and Ω_2^ϵ denote union of the inner elastic inclusions. Γ^ϵ is the interface between Ω_1^ϵ and Ω_2^ϵ and $\partial\Omega$ the boundary of Ω. They are assumed regular.

The elastic structure, characterized by coefficients $a^\epsilon_{ijkh}(x)$ is space-periodic. More precisely, these coefficients derive from the functions:

$$y \to a_{ijkh}(y)$$

defined on the basic period Y, extended by periodicity to $\mathbf{R}^3$, and such that:

$$a_{ijkh} \in L^\infty(\mathbf{R}_3) \tag{1}$$

$$a_{ijkh} = a_{khij} = a_{jikh} \quad \text{(symmetry condition)} \tag{2}$$

$$\exists C_0 > 0;\ a_{ijkh}\xi_{ij}\xi_{kh} \geq C_o\xi_{ij}\xi_{ij} \quad \forall\ \xi_{ij} = \xi_{ji}\dagger \tag{3}$$

(ellipticity condition)

Then the elastic coefficients, defined by

$$a^\epsilon_{ijkh}(x) = a_{ijkh}(x/\epsilon)$$

are ϵY-periodic.

(b) *The equations of the problem*

If the structure is in equilibrium with a system of given external density of body forces $f = (f_i)$, the displacements $u^\epsilon = (u_i^\epsilon)$ being zero on $\partial\Omega$, the stress-tensor $\sigma = (\sigma_{ij})$ and the

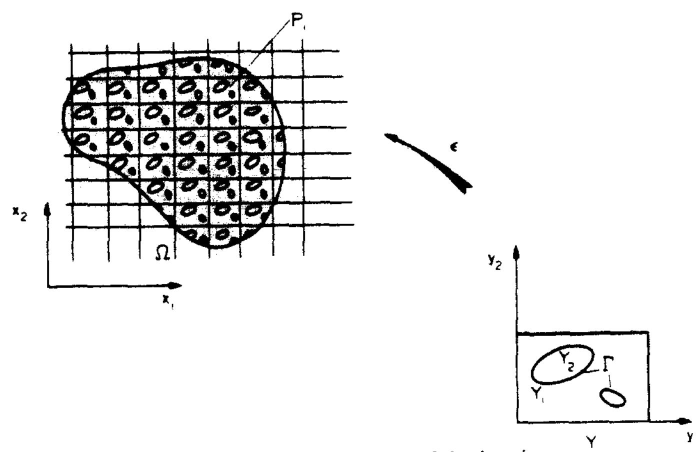


Fig. 1. Periodic geometry of the domain.

†Note that in (3) and in further expressions the usual summation convention is used.

displacement field u satisfy the following relations:

- the equilibrium equations:

$$\frac{\partial \sigma_{ij}^{\epsilon}}{\partial x_j} + f_i = 0 \qquad \text{on } \Omega \tag{4}$$

- the constitutive relation:

$$\sigma_{ij}^{\epsilon} = a_{ijkh}^{\epsilon} \epsilon_{kh}(u^{\epsilon}) \text{ on } \Omega_1^{\epsilon} \text{ and } \Omega_2^{\epsilon},$$

where

$$\epsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5}$$

- the boundary condition:

$$u^{\epsilon} = 0 \qquad \text{on } \partial\Omega \tag{6}$$

- the interface conditions between inclusions and matrix: we assume that the contact holds with only a tangential slip allowed which is "elastic" and characterized by a coefficient $k > 0$, so that:

$$[u_N^{\epsilon}] = 0 \quad [\sigma_N^{\epsilon}] = 0 \quad \sigma_{T1}^{\epsilon} = \sigma_{T2}^{\epsilon} = k/\epsilon[u_T^{\epsilon}] \quad \text{on } \Gamma^{\epsilon} \tag{7}$$

with:

$$u_N = u \cdot n, \qquad u_T = u - u_N n$$
$$\sigma_N = \sigma_{ij} n_i n_j, \qquad (\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i,$$

where n is the outward unit normal to Ω_1^{ϵ}, defined on Γ^{ϵ}, and where the bracket denotes the jump of a function through Γ^{ϵ}

$$[v] = v_2 - v_1.$$

Remarks

- Hypothesis on the body forces and on the boundary conditions allow to state a well-posed problem in terms of the unknown u^{ϵ}. However, they do not occur in further considerations about the homogenized constitutive law which holds in any case of data.
- The constitutive equation (7) is obtained as the limit behavior of a material which elasticity moduli depend on the thickness of the layer and vanish as the thickness decreases to zero. See [6] and [7].

(c) *Variational formulation*

By classical methods, we obtain a variational formulation of the above problem. First, we introduce:

$$V_0 = \{v/v \in (H^1(\Omega))^3,\ v|_{\partial\Omega} = 0\}\dagger \tag{8}$$

$$V^{\epsilon} = \{v = (v_1, v_2),\ v_1 \in [H^1(\Omega_1^{\epsilon})]^3,\ v_2 \in [H^1(\Omega_2^{\epsilon})]^3,\ \ [v_N] = 0 \text{ on } \Gamma^{\epsilon}\} \tag{9}$$

and

$$V_0^{\epsilon} = \{v/v \in V^{\epsilon},\ v|_{\partial\Omega} = 0\} \tag{10}$$

which represents the set of admissible displacement fields. Then, by multiplying (4) by $v \in V_0^{\epsilon}$

†$H^1(\Omega)$ is the Sobolev space defined by:

$$H^1(\Omega) = \left\{v/v \in L^2(\Omega); \frac{\partial v}{\partial x_i} \in L^2(\Omega),\ \forall i = 1,3\right\} \text{ (see [8]).}$$

and integrating by part over Ω we obtain (by applying the Green formula):

$$a^\epsilon(u, v) + \int_{\Gamma^\epsilon} [\sigma_{ij} n_j v_i] \, d\Gamma = L(v)$$

where:

$$a^\epsilon(u, v) = \int_{\Omega_1^\epsilon} a^\epsilon_{ijkh} \epsilon_{kh}(u) \epsilon_{ij}(v) \, dx + \int_{\Omega_2^\epsilon} a^\epsilon_{ijkh} \epsilon_{kh}(u) \epsilon_{ij}(v) \, dx$$

$$L(v) = \int_\Omega f_i v_i \, dx.$$

But, taking (7)–(10) into account, it comes:

$$\int_{\Gamma^\epsilon} [\sigma_{ij} n_j v_i] \, d\Gamma = \int_{\Gamma^\epsilon} \{[\sigma_T v_T] + [\sigma_N] v_N\} \, d\Gamma = \int_{\Gamma^\epsilon} \sigma_T [v_T] \, d\Gamma = k/\epsilon \int_{\Gamma^\epsilon} [u_T][v_T] \, d\Gamma.$$

Therefore, the formulation of the problem can be written out:

$$\left.\begin{aligned} & u^\epsilon \in V_0^\epsilon \\ & a^\epsilon(u, v) + k/\epsilon \, b^\epsilon(u, v) = L(v) \qquad \forall v \in V_0^\epsilon \end{aligned}\right\} \tag{11}$$

with

$$b^\epsilon(u, v) = \int_{\Gamma^\epsilon} [u_T][v_T] \, d\Gamma. \tag{12}$$

and we can state the theorem:

Theorem

Under the assumptions (1)–(3), $f_i \in L^2(\Omega)$ and $k > 0$, there exists, for any $\epsilon > 0$, a unique u^ϵ solution of (11).

The proof of this result is given in [7].

The computer requirements to solve the u^ϵ-problem increase drastically as ϵ decreases to zero since it is essential to have a more and more refined grid to approach accurately the geometry of the inclusions. Therefore, it is natural to analyse the problem in terms of asymptotic expansions and consider the limit state as ϵ decreases to zero.

3. ASYMPTOTIC EXPANSION

We look for an asymptotic expansion of u^ϵ as follows[1]:

$$u^\epsilon(x) = u^0(x, y) + \epsilon u^1(x, y) + \epsilon^2 u^2(x, y) + \cdots /_{y = x/\epsilon} \tag{13}$$

where the functions u^i are Y-periodic with respect to y.

u^ϵ is considered as a function of two scale variables. x expresses the macro effects of the entire structure and y expresses the micro effects of the periodic cell.

Consequently, σ^ϵ_{ij} can be expanded as:

$$\sigma^\epsilon_{ij} = 1/\epsilon \, \sigma^0_{ij} + \sigma^1_{ij} + \epsilon \sigma^2_{ij} + \cdots$$

with

$$\sigma^0_{ij} = a_{ijkh}(y) e_{kh}(u^0), \; e_{kh}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right) \tag{14}$$

$$\sigma^1_{ij} = \Sigma^1_{ij} + a_{ijkh}(y)\epsilon_{kh}(u^0), \Sigma^1_{ij} = a_{ijkh}(y)\epsilon_{kh}(u^1) \tag{15}$$

$$\sigma^2_{ij} = \Sigma^2_{ij} + a_{ijkh}(y)\epsilon_{kh}(u^1), \Sigma^2_{ij} = a_{ijkh}(y)\epsilon_{kh}(u^2). \tag{16}$$

Therefore, problems (P_0) (P_1) . . . arise by identifying the successive ϵ powers in the problem (4)–(7):

(a) ***Problem*** (P_0)

$u^0(x, y)$ is Y-periodic with respect to y and satisfies:

$$-\frac{\partial}{\partial y_j}\sigma^0_{ij} = 0 \qquad \text{in } Y$$

$$\sigma^0_{ij} = a_{ijkh}(y)\epsilon_{kh}(u^0) \qquad \text{in } Y_1 \text{ and } Y_2$$

$$[u^0_N] = 0, \quad [\sigma^0_N] = 0, \quad \sigma^0_{T1} = \sigma^0_{T2} = k[u^0_T] \qquad \text{in } \Gamma.$$

The above problem settled in terms of the variable y is a problem (P) (see Appendix A). Obviously its solution is:

$$u^0 = u^0(x) \tag{17}$$

since x plays the role of a parameter.

One can already note that the first term of the expansion of u does not depend on the micro variable y and can be considered as a mean displacement altered only by higher order terms. However, it is not so for the stress field, the micro variable y is effective from the first term.

(b) ***Problem*** (P_1)

Taking (17) into account, we find that $u^1(x, y)$ is Y-periodic with respect to y and satisfies:

$$-\frac{\partial}{\partial y_j}\Sigma^1_{ij} = \frac{\partial}{\partial y_j}(a_{ijkh})\,\epsilon_{kh}(u^0) \qquad \text{in } Y$$

$$\Sigma^1_{ij} = a_{ijkh}\epsilon_{kh}(u^1) \qquad \text{in } Y_1 \text{ and } Y_2$$

$$\left.\begin{array}{l} [u_N^1] = 0, [\Sigma_N^1] = -[a_{ijkh}]\epsilon_{kh}(u^0)n_i n_j \\ \Sigma^1_{T1} - k[u_T^1] = -(a_{ijkh}\epsilon_{kh}(u^0)n_j)_{T1} \\ \Sigma^1_{T2} - k[u_T^2] = -(a_{ijkh}\epsilon_{kh}(u^0)n_j)_{T2} \end{array}\right\} \text{ on } \Gamma.$$

This formulation shows that $u^1(x, y)$ may be written as:

$$u^1 = -\chi^{pq}(y)\epsilon_{pq}(u^0) \tag{18}$$

the functions χ^{pq} being Y-periodic and such that:

$$-\frac{\partial}{\partial y_j}(\xi^{pq}_{ij}) = \frac{\partial}{\partial y_j}(a_{ijpq}) \text{ in } Y \tag{19}$$

where

$$\xi^{pq}_{ij} = a_{ijkh}\epsilon_{kh}(\chi^{pq}) \qquad \text{in } Y_1 \text{ and } Y_2 \tag{20}$$

$$\left.\begin{array}{l} [\chi_N^{pq}] = 0, [\xi_N^{pq}] = [A_N^{pq}] \qquad \text{with} A_i^{pq} = a_{ijpq} n_j \\ \xi_{T1}^{pq} - k[\chi_T^{pq}] = A_{T1}^{pq} \\ \xi_{T2}^{pq} - k[\chi_T^{pq}] = A_{T2}^{pq} \end{array}\right\} \text{ on } \Gamma. \tag{21}$$

This above problem is also a problem (P) and one can verify that the condition (A10) (see Appendix A) is satisfied. The problems (19)–(21) have solutions which are unique, up to a constant vector, so u^1 is obtained, when u^0 is known, by formula (18).

(c) *Problem* (P_2)

Taking into account the expression (18), $u^2(x, y)$ is a Y-periodic function with respect to y which have to satisfy:

$$-\frac{\partial}{\partial y_j}\Sigma_{ij}^2 = f_i + \frac{\partial}{\partial y_j}\hat{\Sigma}_{ij}^1 + a_{ijkh}\frac{\partial}{\partial x_j}(\epsilon_{kh}(u^1)) + a_{ijkh}\frac{\partial}{\partial x_j}(\epsilon_{kh}(u^0)) \qquad \text{in } Y$$

where

$$\left.\begin{array}{l} \Sigma_{ij}^2 = a_{ijkh}\epsilon_{kh}(u^2) \\ \hat{\Sigma}_{ij}^1 = a_{ijkh}\epsilon_{kh}(u^1) \end{array}\right\} \quad \text{in } Y_1 \text{ and } Y_2$$

$$\left.\begin{array}{l} [u_n^2] = 0, \quad [\Sigma_N^2] = -[\hat{\Sigma}_N^1] \\ \Sigma_{T1}^2 - k[u_T^2] = -\hat{\Sigma}_{T1}^1, \quad \Sigma_{T2}^2 - k[u_T^2] = -\hat{\Sigma}_{T2}^1 \end{array}\right\} \text{ on } \Gamma \tag{22}$$

This problem appears also as a problem (P) (Appendix A). So it has a solution if and only if the condition:

$$\mathscr{L}(c) = 0$$

is satisfied for any constant vector c. With the notations used in the appendix, we calculate:

$$I = \int_y \bar{f}_i c_i \, dy = \int_y c_i f_i \, dy + \int_y c_i \left\{\frac{\partial}{\partial y_j}\hat{\Sigma}_{ij}^1 + a_{ijkh}\frac{\partial}{\partial x_j}\epsilon_{kh}(u^1)) + a_{ijkh}\frac{\partial}{\partial x_j}(\epsilon_{kh}(u^0))\right\} dy.$$

Taking (18) into account, it comes:

$$\begin{aligned} I &= c_i f_i |Y| - \int_\Gamma [\hat{\Sigma}_{ij}^1 n_j] c_i \, d\Gamma - \left(\sum_{\alpha=1}^2 \int_{Y_\alpha} c_i \xi_{ij}^{kh}(y) \, dy\right)\frac{\partial}{\partial x_j}\epsilon_{kh}(u^0) + \left(\int_Y a_{ijkh} \, dy\right)\frac{\partial}{\partial x_j}\epsilon_{kh}(u^0) \\ &= c_i f_i |Y| - \int_\Gamma \{[\hat{\Sigma}_T^1] c_T + [\hat{\Sigma}_N^1] c_N\} \, d\Gamma - \frac{\partial}{\partial x_j}(\epsilon_{kh}(u^0))\left(\sum_{\alpha=1}^2 c_i \int_{Y_\alpha} (\xi_{ij}^{kh} - a_{ijkh}) \, dy\right). \end{aligned}$$

Moreover:

$$\int_\Gamma \{h c_N + (g_2 - g_1) c_T\} \, d\Gamma = -\int_\Gamma \{[\hat{\Sigma}_N^1] c_N + [\hat{\Sigma}_T^1] c_T\} \, d\Gamma.$$

Therefore, $\mathscr{L}(c) = 0$ is satisfied if and only if:

$$f_i |Y| = \left(\sum_{\alpha=1}^2 \int_{Y_\alpha} (\xi_{ij}^{kh} - a_{ijkh}) \, dy\right)\frac{\partial \epsilon_{kh}(u^0)}{\partial x_j} \tag{23}$$

we set:

$$Q_{ijkh}=\frac{1}{|Y|}\int_Y (a_{ijkh}-a_{ijpq}e_{pq}(\chi^{kh}))\,dy \tag{24}$$

then, the eqn (23) can be considered as the static equilibrium equation in the unknown displacement field u^0. The corresponding medium is homogeneous but non-isotropic, its elastic moduli are given by relations (24).

The coefficient Q_{ijkh} are called "homogenized" or "equivalent" moduli; these notations can be justified by the following remark: u^ϵ converges weakly to u^0 – [7] – and therefore, one can consider u^0 as the limit solution of the problem as $\epsilon\to 0$.

4. THE HOMOGENIZED MODULI

To obtain the homogenized moduli Q_{ijkh}, it is necessary, first, to compute the functions $\chi^{pq}(y)$. The integration (24) on Y, then. completes the explicit determination.

(a) *Determination of the functions* $\chi^{pq}(y)$

These functions are solutions of elliptic boundary value problems (19)–(21), on the basic period Y.

As our purpose is to compute by a finite element method the functions χ^{pq}, it necessitates to exhibit the variational formulation of (19)–(21). Using the method worked out in the Appendix, we obtain for this problem:

$$\left.\begin{aligned}&\chi^{pq}\in V\\ &a(\chi^{pq},\varphi)+kb(\chi^{pq},\varphi)=\int_y a_{ijpq}e_{ij}(\varphi)\,dy,\ \forall\varphi\in V\end{aligned}\right\} \tag{25}$$

where

$$a(\varphi,\psi)=\int_{y_1}a_{ijkh}(y)e_{kh}(\varphi)e_{ij}(\psi)\,dy+\int_{y_2}a_{ijkh}(y)e_{kh}(\varphi)e_{ij}(\psi)\,dy$$

$$b(\varphi,\psi)=\int_\Gamma[\varphi_T][\psi_T]\,d\Gamma.$$

The second part of the next section will be devoted to the discretization of equation (25).

Another way to write out the r.h.s. term of (25) is to introduce the functions P^{ij}:

$$\left.\begin{aligned}&P^{ij}=(P^{ij}_1,P^{ij}_2,P^{ij}_3)\\ &P^{ij}_k=y_j\delta_{ik}\quad k=1,2,3\end{aligned}\right\} \tag{26}$$

(25) becomes:

$$\left.\begin{aligned}&\chi^{pq}\in V\\ &a(\chi^{pq},\varphi)+kb(\chi^{pq},\varphi)=a(P^{pq},\varphi),\ \forall\varphi\in V\end{aligned}\right\} \tag{27}$$

and the last expression $a(P^{pq},\varphi)$ can be written in terms of surface load

$$a(P^{pq},\varphi)=-\int_\Gamma[[\lambda\varphi_l]\nu_l\delta_{ij}+[\mu\,\varphi_l](\nu_j\delta_{li}+\nu_i\delta_{lj})]\,d\Gamma. \tag{28}$$

(b) *The effective coefficients*

Provided the numerical computation of the functions χ^{ij}, the Q_{ijkh} derive from the integration (24). (24) shows clearly that each coefficient Q_{ijkh} is the mean value, on Y, of the corresponding a_{ijkh}, altered by a corrective term depending on the χ^{ij}. Furthermore, as k tends to infinity, the solutions $\chi^{ij}(k)$ tend to $\chi^{ij}(\infty)$ solutions of the homogenization problem of the perfectly cohesive case [4, 7]. Hence: $Q_{ijkh}(k) \to Q_{ijkh}(\infty)$.

Moreover, with regards to (26) and (27), we have:

$$Q_{ijkh} = \frac{1}{|Y|} a(P^{kh} - \chi^{kh}, P^{ij})$$

and

$$a(\chi^{kh}, \chi^{ij}) + kb(\chi^{kh}, \chi^{ij}) = a(P^{kh}, \chi^{ij}),$$

therefore:

$$Q_{ijkh} = \frac{1}{|Y|} a(P^{kh} - \chi^{kh}, P^{ij} - \chi^{ij}) + kb(\chi^{kh}, \chi^{ij}).$$

This above relation shows that the symmetry properties of the constitutive law of the initial components remain valid for the equivalent constitutive law.

5. NUMERICAL APPLICATION

(a) *The fiber reinforced material*

The computation of the equivalent coefficients are performed for a composite material consisting of glass fibers coated with resin. Each component is assumed elastic, homogeneous and isotropic, with:

$$E_f = 84\ 10^9\ \text{Pa}; \quad \nu_f = 0.22 \text{ for the fiber}$$

$$E_m = 4\ 10^9\ \text{Pa}; \ \nu_m = 0.34 \text{ for the matrix.}$$

The circular cross section fibers are inserted in a parallel direction to $0x_3$. They are uniformly distributed in the resin with a square basic period. The computation of the Q_{ijkh} moduli is performed for various values of the slip coefficient k and for various fiber-matrix ratios. Problems (27) reduce to two dimensional problems. In a parallel direction to the fibers the material can be considered as periodic but with an arbitrary period and consequently the periodic functions are independent of y_3:

$$\chi^{ij}(y) = \chi^{ij}(y_1, y_2) \qquad i, j = 1, 2, 3.$$

Moreover, we have (see [7]):

$$\chi^{11} = (\chi_1^{11}, \chi_2^{11}, 0)$$

and the same property for χ^{12}, χ^{22} and χ^{33}. These functions are therefore solution of plane strain elasticity problems, with tangential discontinuities through Γ.

For the same reasons, we have:

$$\chi^{13} = (0, 0, \chi_3^{13})$$

and the same property for χ^{23}. These two functions are solution of a scalar problem in $\mathbf{R}^2$, with discontinuities through Γ. In both cases, static or thermal loads are imposed on Γ.

The Q_{ijkh} matrix is symmetric as a consequence of the definition of the coefficients. Moreover, the particularization of a direction $(0, x_3)$, the symmetry of the basic frame (circular cross section fibers and square cells) and some parity properties [4] of the χ^{ij} imply that the equivalent material is orthotropic and the stress–strain relation can be written out:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} Q_{11\,11} & Q_{11\,22} & Q_{11\,33} & & & \\ & Q_{11\,11} & Q_{11\,33} & & 0 & \\ & & Q_{33\,33} & & & \\ & & & 2Q_{13\,13} & & \\ & & & & 2Q_{13\,13} & \\ & & & & & 2Q_{12\,12} \end{Bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{Bmatrix}$$

Numerical results are expressed in terms of Young moduli, Poisson ratios and shear moduli, considering the strain–stress relation:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{Bmatrix} = \begin{Bmatrix} 1/E_1 & -\nu_{12}/E_1 & -\nu_{13}/E_1 & & & \\ & 1/E_1 & -\nu_{13}/E_1 & & & \\ & & 1/E_3 & & & 0 \\ & & & \dfrac{1}{2G_{13}} & & \\ & & & & \dfrac{1}{2G_{13}} & \\ & & & & & \dfrac{1}{2G_{12}} \end{Bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}$$

(b) *Numerical results*

The computation of the χ^{ij} has been performed using the finite element method through the library MODULEF[9]. A classical P_1 interpolation in $H^1(Y^{12})$ with an extension to 0 in Y_2^{12}, respectively a P_1 interpolation in $H^1(Y_2^{12})$ with an extension to 0 in Y_1^{12}, yield an internal approximation to the space V. The displacement fields thus obtained are completely discontinuous through Γ. The continuity condition of the normal component of the displacement fields and the periodicity conditions are performed by a linkage of the concerned degrees of freedom. These elements do not take into account the slip conditions. Therefore, it is essential to insert, along Γ, singular elements, with a 0 thickness, having nevertheless 4 degrees of freedom (for the scalar problems) or 8 degrees of freedom (for the plane strain problems) (see Fig. 2). They allow to compute the jump terms of the variational formulation (27). Figure 3 shows the grid used to calculate functions χ^{ij} on the basic period.

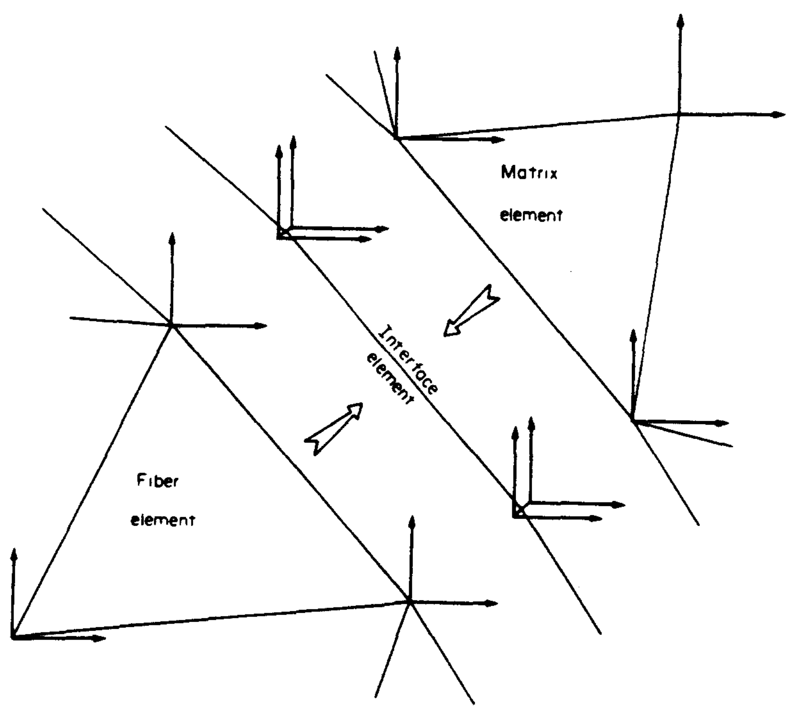


Fig. 2. Assembly of the interface elements.

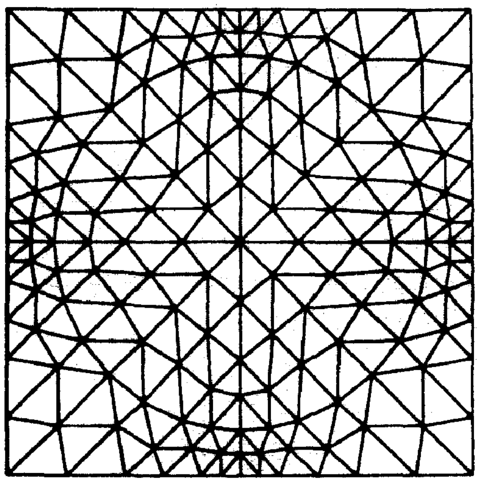
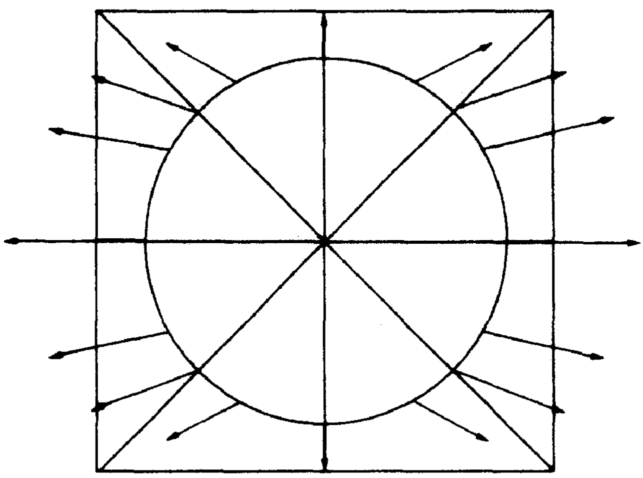
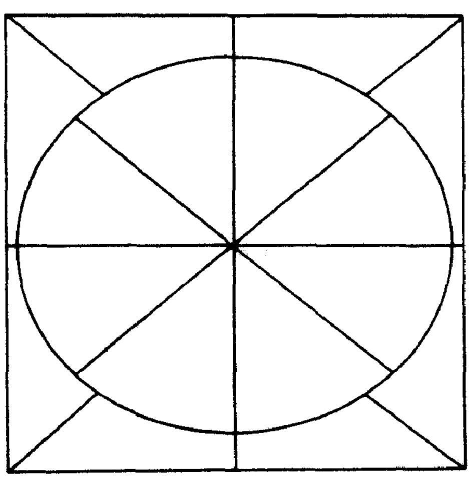


Fig. 3.

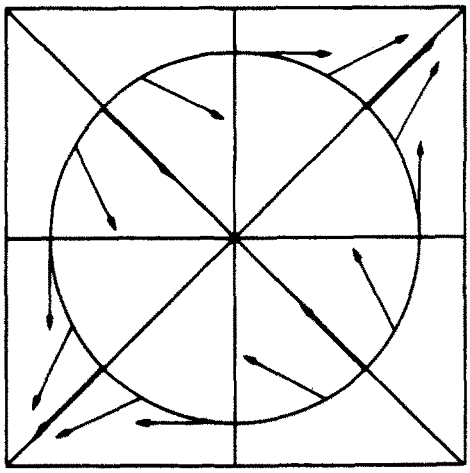


(a)

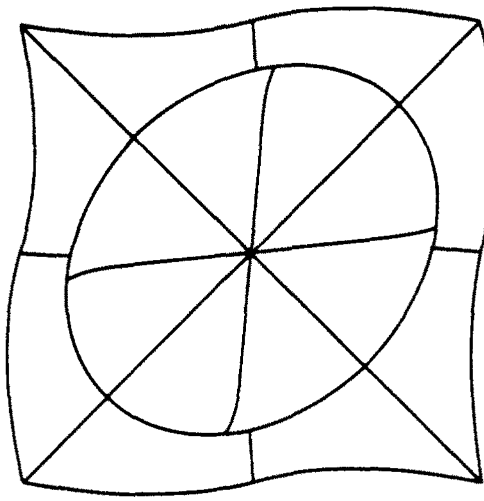


(b)

Fig. 4.



(c)



(d)

Fig. 4. Solutions χ^{ij}; (a) χ^{11} static load; (b) χ^{11}; (c) χ^{12} static load; (d) χ^{12}.

In Fig. 4, (a) and (c) present the static loads (see (28)) applied on Γ to compute χ^{11} and χ^{12}, (b) and (d) show the respective distorted cells, for $k = 10^7$ Pa/m. The slip which occurs along Γ appears sharply on these figures.

The next figures show the alterations of the elastic coefficients of the equivalent material as the damage coefficient:

$$D = \log\left(\frac{10^4}{\kappa} + 1\right)$$

decreases. The coefficient $\kappa = (kL/E_f)$ is dimensionless, the characteristic length L being equal to one.

The Young modulus E_3 and the Poisson ratios $\nu_{13} = \nu_{23}$ do not appear on the Figure: they remain constant and equal to that of the perfectly cohesive case.

When $D = 0$, no slip is allowed, and the coefficients values agree with the classical result [10] (i.e. in the cohesive case). This undamaged state remains until D reaches a lower bound D_m. On the other hand, when D is greater than an upper bound D_M, the coefficients no longer depend on D. The material is in a damaged state and some of the coefficients are then lower than that of

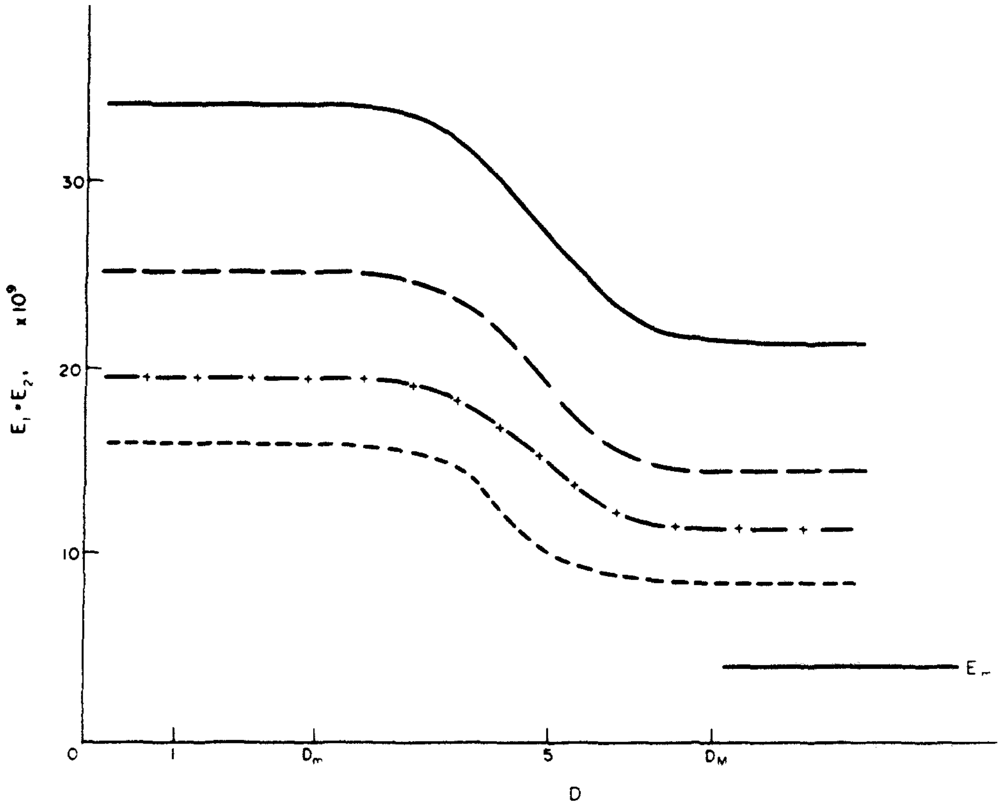

30
20
10
0
1
D_m
5
D_M
D
$E_1 = E_2$, x 10^9
E_m

Fig. 5(a).

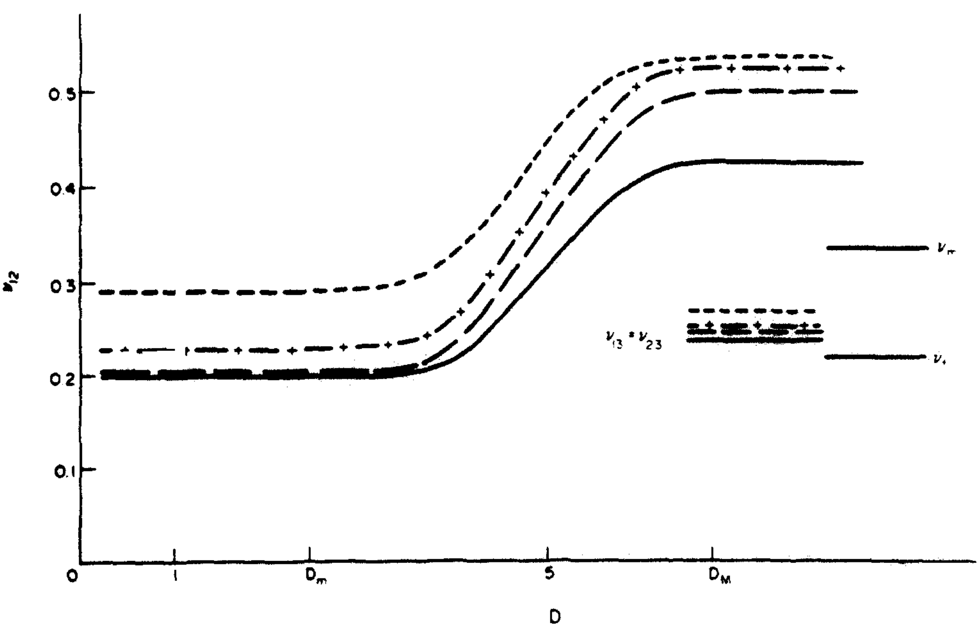

0.5
0.4
0.3
0.2
0.1
0
1
D_m
5
D_M
D
ν_{12}
$\nu_{13} = \nu_{23}$
ν_m
ν_f

Fig. 5(b).

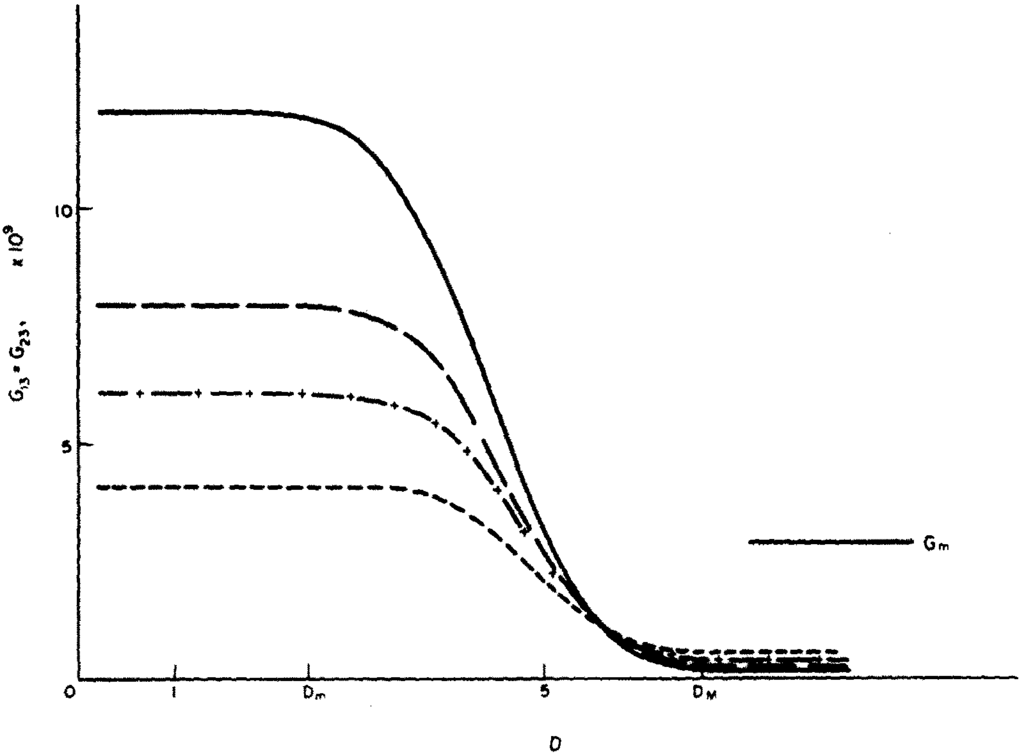


Fig. 5(c).

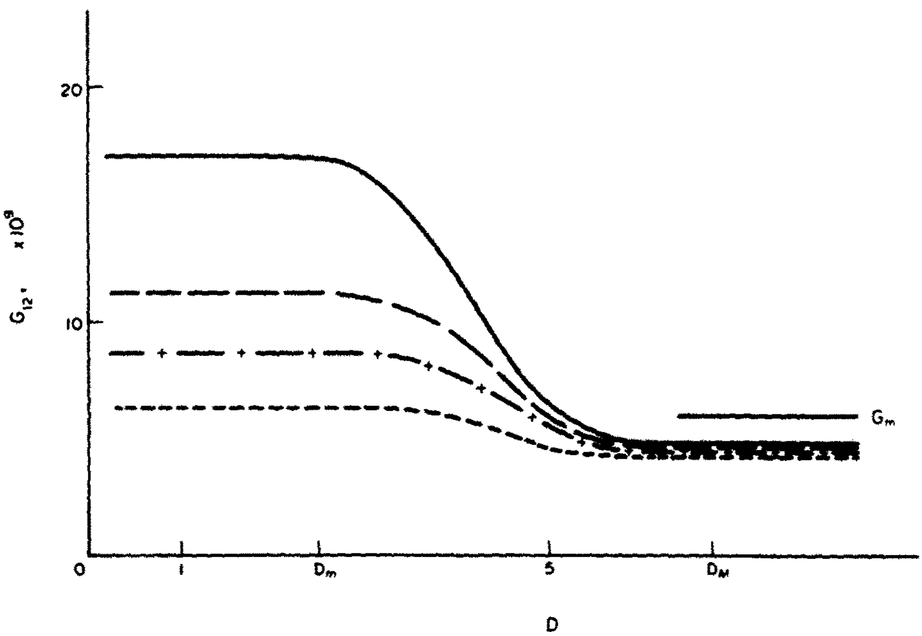


Fig. 5(d).

Fig. 5. (a) Young moduli E_l, E_2; (b) Poisson ratio ν_{12}; (c) Shear moduli G_{13}, G_{23}; (d) Shear modulus G_{12}; matrix ratio (vol. matrix/total vol.); ——— 24%; — — — 30%; + – + – 36%; - - - - - - - 50%.

the matrix (which is a poor elastic material). Particularly, one can point out that the shear moduli G_{13} and G_{23} nearly vanish as D increases.

Another remark is that the steps D_m and D_M are independent of the fiber-matrix ratio and are the same for each coefficient.

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APPENDIX A

Problems (P_0), (P_1) and (P_2) derive from a more general one whose formulation is as follows:

Problem (P)

Find $\hat{u}(y)$, a Y-periodic function satisfying:

$$-\frac{\partial}{\partial y_j}\hat{\sigma}_{ij} = \bar{f}_i(y) \text{ in } Y \tag{A1}$$

$$\hat{\sigma}_{ij} = a_{ijkh}(y)\mathbf{e}_{kh}(u) \quad \text{in } Y_1 \text{ and } Y_2 \tag{A2}$$

$$\left.\begin{aligned} &[\hat{u}_N] = 0, \quad [\hat{\sigma}_N] = h(y) \\ &\hat{\sigma}_{T1} - k[\hat{u}_T] = g_1(y), \quad \hat{\sigma}_{T2} - k[\hat{u}_T] = g_2(y) \end{aligned}\right\} \text{ on } \Gamma \tag{A3}$$

where $\bar{f}$, h, g_1 and g_2 are given functions defined on Y or Γ. We study here this auxiliary problem, and first we give the *variational formulation*.

Let V be:

$$V = \{v = (v_1, v_2), v_1 \in [H^1(Y_1)]^3, v_2 \in [H^1(Y_2)]^3, [v_N] = 0 \text{ on } \Gamma,$$

$$\text{values of } v \text{ being equals on two opposite faces of } Y\}. \tag{A4}$$

Multiplying (A1) by $v \in V$ and integrating over Y, we obtain (by applying Green formula on Y_1 and Y_2):

$$-\int_{\partial Y_1}\hat{\sigma}_{ij}n_jv_i\,\mathrm{d}\Gamma + \int_{\partial Y_2}\hat{\sigma}_{ij}n_jv_i\,\mathrm{d}\Gamma + \int_{Y_1}\hat{\sigma}_{ij}\mathbf{e}_{ij}(v)\,\mathrm{d}y + \int_{Y_2}\hat{\sigma}_{ij}\mathbf{e}_{ij}(v)\,\mathrm{d}y = \int_Y \bar{f}_iv_i\,\mathrm{d}y.$$

Taking into account periodic conditions for v and $[v] = 0$ on Γ, it comes:

$$\int_\Gamma ([\hat{\sigma}_N]v_N + [\hat{\sigma}_Tv_T])\,\mathrm{d}\Gamma + a(\hat{u}, v) = \bar{L}(v)$$

where

$$a(u, v) = \int_{Y_1} a_{ijkh}(y)\mathbf{e}_{kh}(u)\mathbf{e}_{ij}(v)\,\mathrm{d}y + \int_{Y_2} a_{ijkh}(y)\mathbf{e}_{kh}(u)\mathbf{e}_{ij}(v)\,\mathrm{d}y \tag{A5}$$

$$\bar{L}(v) = \int_Y \bar{f}_iv_i\,\mathrm{d}y. \tag{A6}$$

Therefore, using relations (A3) on Γ, (A6) becomes:

$$\int_\Gamma \{h\,v_N + k[\hat{u}_T][v_T] + g_2v_{T2} - g_1v_{T1}\}\,\mathrm{d}\Gamma + a(\hat{u}, v) = \bar{L}(v).$$

We set:

$$b(u, v) = \int_\Gamma [u_T][v_T]\,\mathrm{d}\Gamma \tag{A7}$$

$$D(v) = \int_\Gamma (-h\,v_N - g_2V_{T2} + g_1v_{T1})\,\mathrm{d}\Gamma \tag{A8}$$

and finally, the variational formulation can be written out:

$$\hat{u} \in V$$

$$a(\hat{u}, v) + kb(\hat{u}, v) = \bar{L}(v) + D(v) \qquad \forall v \in V \tag{A9}$$

A necessary condition for the existence of a solution to problem (A9) arises from this formulation. If we choose $v = c$ where c belongs to $\mathscr{C}$ (the set of constant vectors on Y) we obtain:

$$0 = \bar{L}(c) + D(c) = \mathscr{L}(c) \qquad \forall c \in \mathscr{C} \tag{A10}$$

therefore, if $\bar{f}$, h, g_1 and g_2 are such that $\mathscr{L}(c) \neq 0$, problem (P) has no solution. Provided this remark, we can now state the existence and uniqueness result:

Theorem

Under the hypothesis (1)–(3), $k > 0$ and $\bar{f} \in (L^2(Y))^3$, $h \in L^2(\Gamma)$, $g_1, g_2 \in (L^2(\Gamma))^2$ satisfying (A10), the problem (P) has a unique solution, up to a constant vector, denoted $\hat{u}$.

Proof

If the condition (A10) holds, we note that the problem (A9) can be expressed on the quotient space $W = V/\mathscr{C}$. In order to obtain the result, it is essential to look for a convenient norm of space W.

First, we remark that V is a subspace of $[H^1(Y_1)]^3 \times [H^1(Y_2)]^3$ which is an Hilbert space for the norm:

$$\|v\| = (\|v_1\|_1^2 + \|v_2\|_2^2)^{1/2} \tag{A11}$$

where $\|\ \|_k$ is the norm in $[H^1(Y_k)]^3$. If $|\ |_k$ denotes the norm in $L^2(Y_k)$, we choose (cf. [11]):

$$\|v\|_k^2 = \sum_{i,j} |e_{ij}(v)|_k^2 + \sum_i |v_i|_k^2.$$

Therefore, one can state (see [4]) that the quantity

$$|||v||| = (\|v_1\|_1^2 + \sum_{i,j} |e_{ij}(v_2)|_2^2 + b(v, v))^{1/2} \tag{A12}$$

is an hibertian norm, equivalent to (A11), on the space V.

Moreover, in [12] it was shown that W is an Hilbert space with respect to:

$$\|v\| = \operatorname*{Inf}_{c \in \mathscr{C}} |||v + c||| \tag{A13}$$

and, furthermore, we shall prove the auxilary lemma:

Lemma

The quantity

$$N(v) = \left(\sum_{i,j} |e_{ij}(v)|_1^2 + \sum_{i,j} |e_{ij}(v)|_2^2 + b(v, v)\right)^{1/2} \tag{A14}$$

is a norm on W, equivalent to quotient norm (A13).

Consequently, if we consider the bilinear form:

$$(u, v) \rightarrow a(u, v) + kb(u, v), \tag{A15}$$

taking into account (1)–(3), $k > 0$, one can prove that (A14) is continuous and coercive on W. Therefore, applying Lax Milgram theorem, we conclude that the problem (P) has a unique solution on W. So, provided the proof of the previous lemma, the existence theorem holds.

Proof of the auxilary lemma

(i) Obviously, we have:

$$\|v\|^2 = \operatorname*{Inf}_{c \in \mathscr{C}} |||v + c|||^2 = N^2(v) + \operatorname*{Inf}_{c \in \mathscr{C}} |v_1 + c|_1^2$$

so:

$$\|v\|^2 \leq N^2(v).$$

(ii) Reciprocally, let us show that there exists a constant C such that:

$$\forall v \in W, \|v\|^2 \leq CN^2(v). \tag{A16}$$

For this purpose, we set P, the orthogonal projection operator from $(L^2(Y_1))^3$ into $\mathscr{C}$ (with respect to the scalar product corresponding to $|\ |_1$); then:

$$\|v\|^2 = N^2(v) + |v_1 - Pv_1|_1^2$$

and, consequently, (A16) is proved if and only if

$$|v_1 - Pv_1|_1^2 \leq C N^2(v), \quad \forall v = (v_1, v_2) \in V. \tag{A17}$$

To prove (A17) we assume that for any positive n one can associate a sequence $v^n = (v_1^n, v_2^n) \in V$ such that:

$$|v_1^n - Pv_1^n|_1^2 = 1,\ N^2(v^n) \leq 1/n.$$

Setting $w^n = (v_1^n - Pv_1^n, v_2^n - Pv_1^n)$, we get:

$$N^2(w^n) \leq 1/n,\ |w_1^n|_1 = 1,\ (w_1^n, c) = 0 \quad \forall c \in \mathscr{C}.$$

Then, $\{w^n\}$ is bounded in V and there exists a subsequence $\{w^\mu = (w_1^\mu, w_2^\mu)\}$ which converges weakly to w_0 in V; hence, w_1^μ converges strongly in $(L^2(Y_1))^3$ (see [13]), so:

$$|w_1^0|_1 = 1. \tag{A18}$$

Moreover, by lower semi-continuity:

$$N^2(w^0) \leq \varliminf N^2(w^\mu) = 0$$

so:

$$N^2(w^0) = 0.$$

This implies:

$$\mathbf{e}_{ij}(w_1^0) = 0,\ \mathbf{e}_{ij}(w_2^0) = 0,\ b(w^0, w^0) = 0,$$

and consequently:

$$w_1^0 = w_2^0 = c \in \mathscr{C}.$$

Furthermore, $(w_1^0, c) = 0$, $\forall c \in \mathscr{C}$. Therefore, $w^0 = 0$. But it is inconsistent with (A18) and this completes the proof of the lemma.